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Relationships Between Classical and Shear Deformation Theories of Axisymmetric Circular Plates

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The exact relationships between the deflections, slopes/rotations, shear forces, and bending moments of a third-order theory and those of the classical theory and the first-order shear deformation theory for axisymmetric bending of isotropic circular plates are developed. The relationships enable one to obtain the solutions of the third-order theory from any known classical or first-order theory solutions of axisymmetric circular plates for any set of boundary conditions and transverse loads. The relationships may also be used to develop finite element models of axisymmetric bending of circular plates according to the first-order and third-order theories.

Nomenclature

A	= area of cross section of the beam
A_{ij}	= extensional stiffnesses
B_{ij}	= extensional-bending coupling coefficients
D_{ij}	= bending stiffnesses
E_1, E_2	= principal elastic moduli in the material coordinates
F_{ij}	= higher-order stiffnesses
G_{13}	= shear modulus
H_{ij}	= higher-order stiffnesses
h	= plate thickness
K_s	= shear correction factor
M_{rr}, $M_{\theta\theta}$	= radial and circumferential bending moments per unit length
$\mathcal{M}$	= moment sum
P_{rr}, $P_{\theta\theta}$	= higher-order radial and circumferential bending moment per unit length
Q_{ij}	= plane stress reduced elastic stiffnesses
Q_r	= transverse shear force per unit length
q	= distributed transverse load per unit length
R_r	= higher-order transverse shear force per unit length
r	= radial coordinate
r_a, r_b	= inner and outer radii of the circular finite element
u_r	= total radial displacement
u_z	= total transverse displacement
V_r	= effective shear force per unit length; see Eq. (12)
w_0	= transverse deflection
z	= transverse coordinate
α	= $4/(3h^2)$
β	= $4/h^2$
γ_{rz}	= transverse shear strain
ε_{rr}, $\varepsilon_{\theta\theta}$	= radial and circumferential strains
θ	= circumferential coordinate
ν_{12}, ν_{21}	= Poisson ratios
ϕ	= rotation of a transverse normal

I. Introduction

THERE are a number of theories that are used to represent the kinematics of deformation of the axisymmetric bending of circular plates. To describe various theories, we introduce the following coordinate system. The r coordinate is taken radially outward from the center of the plate, the z coordinate is taken along the thickness (or height) of the plate, and the θ coordinate is taken along a circumference of the plate. In a general case where applied loads and geometric boundary conditions are not axisymmetric, the displacements (u_r, u_θ, u_z) along the coordinates (r, θ, z) are functions of r, θ, and z coordinates. Here we assume that the applied loads and boundary conditions are independent of θ coordinate, i.e., axisymmetric, so that the displacement u_θ is identically zero and (u_r, u_z) are only functions of r and z.

The most commonly used and the simplest theory is the classical plate theory (CPT), which is based on the Kirchhoff hypothesis that straight lines normal to the midplane before deformation remain 1) inextensible, 2) straight, and 3) normal to the midsurface after deformation. The Kirchhoff hypothesis amounts to neglecting both transverse shear and transverse normal effects, i.e., deformation is due entirely to bending and in-plane stretching.

The first-order shear deformation plate theory (FST) (see, for example, Refs. 1 and 2) is the simplest theory that accounts for nonzero transverse shear strain. The first-order theory includes a constant state of transverse shear strain with respect to the thickness coordinate and, hence, requires shear correction factors, which depend not only on the material and geometric parameters but also on the loading and boundary conditions. Second- and higher-order theories use higher-order expansions of the displacement components through the thickness of the plate. They further relax the Kirchhoff hypothesis by removing the assumption of straightness of a transverse normal. In all theories the inextensibility of transverse normals can be removed by assuming that the transverse deflection also varies through the thickness.

The third-order plate theory of Reddy[1,2] is based on the displacement field

$$u_r(r, z) = z\phi(r) - \alpha z^3\left(\phi + \frac{\mathrm{d}w_0}{\mathrm{d}r}\right), \qquad u_z(r, z) = w_0(r) \tag{1}$$

The displacement field accommodates quadratic variation of transverse shear strains (and hence stresses) and vanishing of transverse shear strain and, hence, shear stress on the top and bottom planes of a plate. Thus there is no need to use shear correction factors in a third-order theory. Levinson[3] used a vector approach to derive the equations of equilibrium, which are essentially the same as those of the Timoshenko theory. Bickford[4] and Reddy[5] independently derived variationally consistent equations of motion associated with the displacement field of the type in Eq. (1). Bickford's work[4] was limited to isotropic beams, whereas Reddy's study considered laminated composite plates. The third-order laminated plate theory of Reddy[5] was specialized by Heyliger and Reddy[6] to study linear and nonlinear bending and vibrations of isotropic beams. For other

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pertinent works on third-order theory of beams and plates, the reader may consult the textbook by Reddy,[2] which contains a complete review of shear deformation plate theories.

The objective of this paper is to develop exact relationships between the axisymmetric bending of circular plates using the classical plate theory and the third-order plate theory based on displacement field (1). Wang[7] developed relationships between the solutions of the Euler–Bernoulli beam theory and the Timoshenko beam theory, and Wang and Lee[8] developed similar relationships for axisymmetric bending of circular plates. Recently, Reddy et al.[9] developed relationships between the bending solutions of the Euler–Bernoulli beam theory and the Reddy third-order beam theory. Here we use a similar approach to develop the relationships for axisymmetric bending of circular plates. The results developed herein enable one to obtain the exact solutions for deflections, bending moments, shear forces of the FST, and Reddy's third-order plate theory (TPT) by knowing the corresponding solutions of the same problem based on the CPT, which can be found in any book on theory of plates.

II. Equations of Reddy's Third-Order Plate Theory

Here we develop the equations of the TPT using the displacement in Eq. (1). The nonzero linear strains can be written as

$$\varepsilon_{rr} = z\varepsilon_{rr}^{(1)} + z^3\varepsilon_{rr}^{(3)}, \qquad \varepsilon_{\theta\theta} = z\varepsilon_{\theta\theta}^{(1)} + z^3\varepsilon_{\theta\theta}^{(3)}$$
$$\gamma_{rz} = \gamma_{rz}^{(0)} + z^2\gamma_{rz}^{(2)} \tag{2}$$

where

$$\varepsilon_{rr}^{(1)} = \frac{d\phi}{dr}, \qquad \varepsilon_{rr}^{(3)} = -\alpha\left(\frac{d\phi}{dr} + \frac{d^2 w_0}{dr^2}\right)$$
$$\varepsilon_{\theta\theta}^{(1)} = \frac{\phi}{r}, \qquad \varepsilon_{\theta\theta}^{(3)} = -\alpha\left(\frac{\phi}{r} + \frac{1}{r}\frac{dw_0}{dr}\right) \tag{3}$$
$$\gamma_{rz}^{(0)} = \phi + \frac{dw_0}{dr}, \qquad \gamma_{rz}^{(2)} = -\beta\left(\phi + \frac{dw_0}{dr}\right)$$

and

$$\alpha = 4/(3h^2), \qquad \beta = 3\alpha = 4/h^2 \tag{4}$$

The principle of virtual displacements may be used to derive the equations of equilibrium.[2] The equations of equilibrium are given by

$$\frac{d}{dr}(r\bar{Q}_r) + \alpha\frac{d^2}{dr^2}(rP_{rr}) - \alpha\frac{dP_{\theta\theta}}{dr} + rq = 0 \tag{5}$$

$$\frac{d}{dr}(r\bar{M}_{rr}) - r\bar{Q}_r - \bar{M}_{\theta\theta} = 0 \tag{6}$$

where

$$\begin{Bmatrix} M_{rr} \\ P_{rr} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} z \\ z^3 \end{Bmatrix} \sigma_{rr}\, dz, \qquad \begin{Bmatrix} M_{\theta\theta} \\ P_{\theta\theta} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} z \\ z^3 \end{Bmatrix} \sigma_{\theta\theta}\, dz \tag{7}$$

$$\begin{Bmatrix} Q_r \\ R_r \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} 1 \\ z^2 \end{Bmatrix} \sigma_{rz}\, dz \tag{8}$$

$$\bar{M}_{rr} = M_{rr} - \alpha P_{rr}, \qquad \bar{M}_{\theta\theta} = M_{\theta\theta} - \alpha P_{\theta\theta}, \qquad \bar{Q}_r = Q_r - \beta R_r \tag{9}$$

The primary and secondary variables of the theory are as follows. Primary variables:

$$w_0, \frac{dw_0}{dr}, \phi \tag{10}$$

Secondary variables:

$$V_r, P_{rr}, \bar{M}_{rr} \tag{11}$$

where

$$rV_r \equiv r\bar{Q}_r + \alpha\left[\frac{d}{dr}(rP_{rr}) - P_{\theta\theta}\right] \tag{12}$$

The specification of a primary variable constitutes a geometric boundary condition, whereas the specification of a secondary variable constitutes a force boundary condition. Note that the present TPT requires the specification of both ϕ and dw_0/dr, and the effective shear force in TPT is V_r.

For polar orthotropic plates with material principal axes coinciding with the plate coordinates, the stress resultants (M_{rr}, P_{rr}, $M_{\theta\theta}$, $P_{\theta\theta}$, Q_r, R_r) can be expressed in terms of the strains by the relations

$$\begin{Bmatrix} M_{rr} \\ M_{\theta\theta} \\ P_{rr} \\ P_{\theta\theta} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & F_{11} & F_{12} \\ D_{12} & D_{22} & F_{12} & F_{22} \\ F_{11} & F_{12} & H_{11} & H_{12} \\ F_{12} & F_{22} & H_{12} & H_{22} \end{bmatrix} \begin{Bmatrix} \varepsilon_{rr}^{(1)} \\ \varepsilon_{\theta\theta}^{(1)} \\ \varepsilon_{rr}^{(3)} \\ \varepsilon_{\theta\theta}^{(3)} \end{Bmatrix} \tag{13}$$

$$\begin{Bmatrix} Q_r \\ R_r \end{Bmatrix} = \begin{bmatrix} A_{44} & D_{44} \\ D_{44} & F_{44} \end{bmatrix} \begin{Bmatrix} \gamma_{rz}^{(0)} \\ \gamma_{rz}^{(2)} \end{Bmatrix} \tag{14}$$

where D_{ij} ($i, j = 1, 2$) are the bending stiffnesses, A_{44} is the transverse shear stiffness, and (F_{ij}, H_{ij}, D_{44}, F_{44}) are the higher-order stiffnesses:

$$(D_{ij}, F_{ij}, H_{ij}) = \int_{-h/2}^{h/2} Q_{ij}(z^2, z^4, z^6)\, dz \tag{15}$$

$$(A_{44}, D_{44}, F_{44}) = \int_{-h/2}^{h/2} Q_{44}(1, z^2, z^4)\, dz \tag{16}$$

where the various Q_{ij} are known in terms of four independent material (engineering) constants E_1, E_2, ν_{12}, and G_{13}:

$$Q_{11} = \frac{E_1}{1 - \nu_{12}\nu_{21}}, \qquad Q_{12} = \frac{\nu_{12}E_2}{1 - \nu_{12}\nu_{21}}$$
$$Q_{22} = \frac{E_2}{1 - \nu_{12}\nu_{21}}, \qquad Q_{44} = G_{13} \tag{17}$$

Poisson's ratio ν_{21} is known through the reciprocal relation $\nu_{21} = E_1\nu_{12}/E_2$.

It is informative to note that at a clamped edge we require $\phi = dw_0/dr = 0$ in TPT. Consequently, the shear force Q_r of CPT and FST (and $\bar{Q}_r$) computed through constitutive equation (14) is zero at a clamped edge in TPT. But the effective shear force V_r [see Eq. (12)] of the TPT is not zero at a clamped edge because dP_{rr}/dr is not zero there.

III. Bending Relationships Between CPT, FST, and TPT

A. Summary of Equations

The bending equations of equilibrium and stress resultant-displacement relations of the three theories are summarized next for constant material and geometric properties.

The CPT:

$$-\frac{d}{dr}\left(rQ_r^C\right) = rq \tag{18a}$$

$$rQ_r^C \equiv \frac{d}{dr}\left(rM_{rr}^C\right) - M_{\theta\theta}^C \tag{18b}$$

$$M_{rr}^C = -D_{11}\frac{d^2 w_0^C}{dr^2} - D_{12}\frac{1}{r}\frac{dw_0^C}{dr} \tag{19a}$$

$$M_{\theta\theta}^C = -D_{12}\frac{d^2 w_0^C}{dr^2} - D_{22}\frac{1}{r}\frac{dw_0^C}{dr} \tag{19b}$$

where Q_r^C denotes the shear force in the CPT.

The FST:

$$-\frac{d}{dr}\left(rM_{rr}^F\right) + M_{\theta\theta}^F + rQ_r^F = 0 \tag{20a}$$

$$-\frac{d}{dr}\left(rQ_r^F\right) = rq \tag{20b}$$

$$M_{rr}^F = D_{11}\frac{d\phi^F}{dr} + D_{12}\frac{1}{r}\phi^F \tag{21a}$$

$$M_{\theta\theta}^F = D_{12}\frac{d\phi^F}{dr} + D_{22}\frac{1}{r}\phi^F \tag{21b}$$

$$Q_r^F = A_{44}K_s\left(\phi^F + \frac{dw_0^F}{dr}\right) \tag{21c}$$

Reddy's TPT:

$$-\frac{d}{dr}\left(rM_{rr}^T\right) + M_{\theta\theta}^T + rQ_r^T + \alpha\left[\frac{d}{dr}(rP_{rr}) - P_{\theta\theta}\right] - \beta rR_r = 0 \tag{22a}$$

$$-\frac{d}{dr}\left(rQ_r^T\right) + \beta\frac{d}{dr}(rR_r) - \alpha\left[\frac{d^2}{dr^2}(rP_{rr}) - \frac{dP_{\theta\theta}}{dr}\right] = rq \tag{22b}$$

$$M_{rr}^T = D_{11}\frac{d\phi^T}{dr} + D_{12}\frac{1}{r}\phi^T - \alpha\left[F_{11}\left(\frac{d\phi^T}{dr} + \frac{d^2w_0^T}{dr^2}\right) + F_{12}\frac{1}{r}\left(\phi^T + \frac{dw_0^T}{dr}\right)\right] \tag{23a}$$

$$M_{\theta\theta}^T = D_{12}\frac{d\phi^T}{dr} + D_{22}\frac{1}{r}\phi^T - \alpha\left[F_{12}\left(\frac{d\phi^T}{dr} + \frac{d^2w_0^T}{dr^2}\right) + F_{22}\frac{1}{r}\left(\phi^T + \frac{dw_0^T}{dr}\right)\right] \tag{23b}$$

$$Q_r^T = \bar{A}_{44}\left(\phi^T + \frac{dw_0^T}{dr}\right) \tag{24}$$

$$P_{rr} = F_{11}\frac{d\phi^T}{dr} + F_{12}\frac{1}{r}\phi^T - \alpha\left[H_{11}\left(\frac{d\phi^T}{dr} + \frac{d^2w_0^T}{dr^2}\right) + H_{12}\frac{1}{r}\left(\phi^T + \frac{dw_0}{dr}\right)\right] \tag{25a}$$

$$P_{\theta\theta} = F_{12}\frac{d\phi^T}{dr} + F_{22}\frac{1}{r}\phi^T - \alpha\left[H_{12}\left(\frac{d\phi^T}{dr} + \frac{d^2w_0^T}{dr^2}\right) + H_{22}\frac{1}{r}\left(\phi^T + \frac{dw_0}{dr}\right)\right] \tag{25b}$$

$$R_r = \bar{D}_{44}\left(\phi^T + \frac{dw_0^T}{dr}\right) \tag{26}$$

where quantities with superscript C refer to the classical plate theory and with F refer to the first-order shear deformation plate theory (K_s denotes the shear correction factor) and quantities with T refer to the third-order plate theory of Reddy. The following notation for stiffnesses is used in Eqs. (24–26) and subsequently:

$$\bar{D}_{ij} = D_{ij} - \alpha F_{ij}, \qquad \bar{F}_{ij} = F_{ij} - \alpha H_{ij}, \qquad \bar{A}_{44} = A_{44} - \beta D_{44}$$
$$\bar{D}_{44} = D_{44} - \beta F_{44}, \qquad \hat{A}_{44} = \bar{A}_{44} - \beta\bar{D}_{44} \tag{27}$$

For the isotropic case, we have [$E_1 = E_2$, $\nu_{12} = \nu_{21} = \nu$, and $G_{12} = G = E/2(1+\nu)$]

$$D_{11} = D_{22} = D, \qquad D_{12} = \nu D, \qquad D = \frac{Eh^3}{12(1-\nu^2)}$$
$$F_{11} = F_{22} = F, \qquad F_{12} = \nu F, \qquad F = \frac{Eh^5}{80(1-\nu^2)}$$
$$H_{11} = H_{22} = H, \qquad H_{12} = \nu H, \qquad H = \frac{Eh^7}{448(1-\nu^2)} \tag{28}$$
$$A_{44} = Gh, \qquad D_{44} = \frac{Gh^3}{12} = \frac{(1-\nu)}{2}D$$
$$F_{44} = \frac{Gh^5}{80} = \frac{(1-\nu)}{2}F$$

B. Relationships Between CPT and FST Solutions

The deflection, bending moment, and shear force of the first-order plate theory can be expressed in terms of the corresponding quantities of the classical plate theory for axisymmetric bending of isotropic circular plates. The relationships are established using the load equivalence.[8,9]

We introduce the moment sum[10]

$$\mathcal{M} \equiv \frac{M_{rr} + M_{\theta\theta}}{1+\nu} \tag{29}$$

Using Eqs. (19a) and (19b) in Eq. (29), we can show that

$$\mathcal{M}^C = -D\left(\frac{d^2w_0^C}{dr^2} + \frac{1}{r}\frac{dw_0^C}{dr}\right) = -D\frac{1}{r}\frac{d}{dr}\left(r\frac{dw_0^C}{dr}\right) \tag{30}$$

and

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\mathcal{M}^C}{dr}\right) = -q \tag{31}$$

We can also establish the following equality using the definition (29) and Eqs. (19a) and (19b):

$$r\frac{d\mathcal{M}^C}{dr} = \frac{d}{dr}\left(rM_{rr}^C\right) - M_{\theta\theta}^C = rQ_r^C \tag{32}$$

Similarly, we have

$$\mathcal{M}^F = D\left(\frac{d\phi^F}{dr} + \frac{1}{r}\phi^F\right) = D\frac{1}{r}\frac{d}{dr}(r\phi^F) \tag{33}$$

and

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\mathcal{M}^F}{dr}\right) = -q \tag{34}$$

$$r\frac{d\mathcal{M}^F}{dr} = \frac{d}{dr}\left(rM_{rr}^F\right) - M_{\theta\theta}^F = rQ_r^F \tag{35}$$

From Eqs. (18a), (18b), and (20b), it follows that

$$rQ_r^F = rQ_r^C + c_1 \tag{36}$$

and from Eqs. (32), (35), and (36), we have

$$r\frac{d\mathcal{M}^F}{dr} = r\frac{d\mathcal{M}^C}{dr} + c_1$$

or

$$\mathcal{M}^F = \mathcal{M}^C + c_1\log r + c_2 \tag{37}$$

where c_1 and c_2 are constants of integration.

Next, from Eqs. (30), (33), and (37), we have

$$\phi^F = -\frac{dw_0^C}{dr} + \frac{c_1 r}{4D}(2\log r - 1) + \frac{c_2 r}{2D} + \frac{c_3}{rD} \tag{38}$$

Finally, from Eqs. (21c), (36), and (38), we obtain

$$\frac{dw_0^F}{dr} = -\phi^F + \frac{1}{GhK_s}\left(Q_r^C + \frac{c_1}{r}\right) \tag{39}$$

and noting that $Q_r^C = d\mathcal{M}^c/dr$, we have

$$w_0^F = w_0^C + \frac{c_1r^2}{4D}(1-\log r) + \frac{c_1}{K_sGh}\log r - \frac{c_2r^2}{4D} - \frac{c_3\log r}{D} + \frac{c_4}{D} + \frac{\mathcal{M}^C}{K_sGh} \tag{40}$$

The four constants of integration are determined using the boundary conditions of the problem. Note that, for a solid circular plate for $r = 0$, Eq. (36) gives $c_1 = 0$.

To illustrate the evaluation of the constants of integration using boundary conditions, we consider axisymmetric bending of a solid circular plate of radius R_0, clamped (or fixed) at its edge, and subjected to a uniformly distributed transverse load, $q = q_0$. The boundary conditions are as follows.

CPT:

$$w_0^C(R_0) = \frac{\mathrm{d}w_0^C}{\mathrm{d}r}(R_0) = \frac{\mathrm{d}w_0^C}{\mathrm{d}r}(0) = 0 \tag{41}$$

FST:

$$w_0^F(R_0) = \phi^F(R_0) = \phi^F(0) = 0 \tag{42}$$

These conditions yield

$$c_2 = 0, \qquad c_3 = 0, \qquad c_4 = -\frac{D}{KGh}\left(\int Q_r^C\,\mathrm{d}r\right)_{R_0} \tag{43}$$

(We already have $c_1 = 0$ for solid circular plates.)

The classical plate theory deflection $w_0^C(r)$ and shear force $Q_r^C(r)$ of a clamped circular plate are given by

$$w_0^C(r) = \frac{q_0 R_0^4}{64D}\left[1 - \left(\frac{r}{R_0}\right)^2\right]^2, \qquad Q_r^C(r) = -\frac{q_0 r}{2} \tag{44a}$$

Hence we have

$$c_4 = \frac{q_0 R_0^2 h^2}{24(1-\nu)K_s} \tag{44b}$$

and the deflection of the plate according to the first-order theory is given by [replace G with $E/2(1+\nu)$]

$$\begin{aligned} w_0^F(r) &= w_0^C(r) + \frac{h^2}{6D(1-\nu)K_s}\int Q_r^F(r)\,\mathrm{d}r + \frac{c_4}{D} \\ &= \frac{q_0 R_0^4}{64D}\left[1 - \left(\frac{r}{R_0}\right)^2\right]^2 + \frac{q_0 R_0^2 h^2}{24D(1-\nu)K_s}\left[1 - \left(\frac{r}{R_0}\right)^2\right] \end{aligned} \tag{45}$$

Clearly, the effect of shear deformation is to increase the deflection by an amount equal to the second term in the preceding equation.

C. Relationships Between Solutions of CPT and TPT

Here we develop the relationships between the bending solutions of CPT and TPT. At the outset, we note that both the classical and first-order plate theories are fourth-order theories, whereas Reddy's third-order plate theory is a sixth-order theory. The order referred to here is the total order of all equations of equilibrium expressed in terms of the generalized displacements. The third-order plate theory is governed by a fourth-order equation in w_0^T and a second-order equation in ϕ^T. Therefore, the relationships between the solutions of two different order theories can only be established by solving an additional second-order equation.

First we note that Eqs. (22a) and (22b) together yield

$$-\frac{\mathrm{d}^2}{\mathrm{d}r^2}\left(rM_{rr}^T\right) + \frac{\mathrm{d}M_{\theta\theta}^T}{\mathrm{d}r} = rq \tag{46}$$

In addition, Eq. (22a) can be written in terms of the effective shear force V_r^T of Eq. (12) as

$$-\frac{\mathrm{d}}{\mathrm{d}r}\left(rM_{rr}^T\right) + M_{\theta\theta}^T + rV_r^T = 0 \tag{47}$$

From Eqs. (12) and (22b), we have

$$-\frac{\mathrm{d}}{\mathrm{d}r}\left(rV_r^T\right) = rq \tag{48}$$

Hence it follows, from Eqs. (18a), (18b), and (48), that

$$rV_r^T = rQ_r^C + c_1 \tag{49}$$

Next we introduce the moment and higher-order moment sums

$$\mathcal{M}^T = \frac{M_{rr}^T + M_{\theta\theta}^T}{(1+\nu)}, \qquad \mathcal{P} = \frac{P_{rr} + P_{\theta\theta}}{(1+\nu)} \tag{50}$$

Using the definitions (50) and Eqs. (23) and (25), one can show that

$$\mathcal{M}^T = \bar{D}\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r\phi^T) - \alpha F\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}w_0^T}{\mathrm{d}r}\right) \tag{51}$$

$$\mathcal{P} = \bar{F}\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r\phi^T) - \alpha H\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}w_0^T}{\mathrm{d}r}\right) \tag{52}$$

$$r\frac{\mathrm{d}\mathcal{M}^T}{\mathrm{d}r} = \frac{\mathrm{d}}{\mathrm{d}r}\left(rM_{rr}^T\right) - M_{\theta\theta}^T \tag{53}$$

$$r\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}r} = \frac{\mathrm{d}}{\mathrm{d}r}(rP_{rr}) - P_{\theta\theta} \tag{54}$$

Substituting for $r\phi^T$ from Eq. (24) into Eqs. (51) and (52), we arrive at

$$r\mathcal{M}^T = \frac{\bar{D}}{\bar{A}_{44}}\frac{\mathrm{d}}{\mathrm{d}r}\left(rQ_r^T\right) - D\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}w_0^T}{\mathrm{d}r}\right) \tag{55}$$

$$r\mathcal{P} = \frac{\bar{F}}{\bar{A}_{44}}\frac{\mathrm{d}}{\mathrm{d}r}\left(rQ_r^T\right) - F\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}w_0^T}{\mathrm{d}r}\right) \tag{56}$$

Now solving Eq. (55) for $(\mathrm{d}/\mathrm{d}r)(r\,\mathrm{d}w_0^T/\mathrm{d}r)$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}w_0^T}{\mathrm{d}r}\right) = -\frac{1}{D}r\mathcal{M}^T + \frac{\bar{D}}{D\bar{A}_{44}}\frac{\mathrm{d}}{\mathrm{d}r}\left(rQ_r^T\right) \tag{57}$$

Substituting the result into Eq. (56), we obtain

$$r\mathcal{P} = \alpha\left(\frac{F^2 - DH}{D\bar{A}_{44}}\right)\frac{\mathrm{d}}{\mathrm{d}r}\left(rQ_r^T\right) + \frac{F}{D}(r\mathcal{M}^T) \tag{58}$$

From Eqs. (32), (47), (50), and (53), we obtain the result

$$\mathcal{M}^T = \mathcal{M}^C + c_1\log r + c_2 \tag{59}$$

Next we use Eqs. (30), (51), and (54) to arrive at

$$\bar{D}\phi^T - \alpha F\frac{\mathrm{d}w_0^T}{\mathrm{d}r} = -D\frac{\mathrm{d}w_0^C}{\mathrm{d}r} + \frac{c_1 r}{4}(2\log r - 1) + \frac{c_2 r}{2} + \frac{c_3}{r} \tag{60}$$

From Eqs. (47) and (12), we have

$$r\left(Q_r^T - \beta R_r\right) = \frac{\mathrm{d}}{\mathrm{d}r}\left(rM_r^T\right) - M_{\theta\theta}^T - \alpha\left[\frac{\mathrm{d}}{\mathrm{d}r}(rP_{rr}) - P_{\theta\theta}\right] \tag{61}$$

Substituting Eqs. (51) and (52) and $R_r = (\bar{D}_{44}/A_{44})Q_r^T$ into Eq. (61), we obtain

$$\begin{aligned} \left(\frac{\hat{A}_{44}}{\bar{A}_{44}}\right)\left(rQ_r^T\right) &= \left(\frac{\bar{D}}{D}\right)r\frac{\mathrm{d}\mathcal{M}^T}{\mathrm{d}r} \\ &\quad - \alpha^2\left(\frac{F^2 - DH}{D\bar{A}_{44}}\right)r\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(rQ_r^T\right)\right] \end{aligned} \tag{62}$$

and using Eqs. (53), (47), and (49), we have

$$\begin{aligned} r\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(rQ_r^T\right)\right] + \frac{D}{\alpha^2}\left(\frac{\hat{A}_{44}}{F^2 - DH}\right)\left(rQ_r^T\right) \\ = \left[\frac{\bar{D}\bar{A}_{44}}{\alpha^2(F^2 - DH)}\right]\left(rQ_r^T + c_1\right) \end{aligned} \tag{63}$$

Thus a second-order equation must be solved to determine the shear force.

Next, we derive the relationships between deflections w_0^T and w_0^C and rotation ϕ^T and slope $-\mathrm{d}w_0^C/\mathrm{d}r$. Replacing $\mathcal{M}^T$ in terms of $\mathcal{M}^C$ by means of Eq. (59), and then using Eq. (30), one can write Eq. (57) as

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}w_0^T}{\mathrm{d}r}\right) = \frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}w_0^C}{\mathrm{d}r}\right) - \frac{r}{D}(c_1\log r + c_2) + \frac{\bar{D}}{D\bar{A}_{44}}\frac{\mathrm{d}}{\mathrm{d}r}\left(rQ_r^T\right) \tag{64}$$

Integrating twice with respect to r, we obtain

$$\frac{\mathrm{d}w_0^T}{\mathrm{d}r} = \frac{\mathrm{d}w_r^C}{\mathrm{d}r} - \frac{1}{D}\left[\frac{c_1 r}{4}(2\log r - 1) + \frac{c_2 r}{2} + \frac{c_3}{r}\right] + \left(\frac{\bar{D}}{D\bar{A}_{44}}\right)Q_r^T \tag{65}$$

$$w_0^T = w_0^C - \frac{1}{D}\left[\frac{c_1 r^2}{4}(\log r - 1) + \frac{c_2 r^2}{4} + c_3\log r + c_4\right] + \left(\frac{\bar{D}}{D\bar{A}_{44}}\right)\int Q_r^T\,\mathrm{d}r \tag{66}$$

Finally, using Eq. (65) in Eq. (60), we obtain

$$\phi^T = -\frac{\mathrm{d}w_0^C}{\mathrm{d}r} + \frac{1}{D}\left[\frac{c_1 r}{4}(2\log r - 1) + \frac{c_2 r}{2} + \frac{c_3}{r}\right] + \left(\frac{\alpha F}{D\bar{A}_{44}}\right)Q_r^T \tag{67}$$

It is informative to discuss various types of boundary conditions in terms of the dependent variables for the third-order theory. Because a second-order equation for Q_r^T must be solved to determine solutions of the third-order theory, it is also useful to have the boundary conditions on Q_r^T for various types of edge supports. These are listed next.

Clamped edge:

$$\phi^T = 0, \qquad \frac{\mathrm{d}w_0^T}{\mathrm{d}r} = 0, \qquad \text{which imply} \qquad Q_r^T = 0 \tag{68a}$$

$$w_0^T = 0 \tag{68b}$$

Simply supported edge:

$$M_{rr}^T = 0, \qquad P_{rr}^T = 0, \qquad \text{which imply} \quad r\frac{\mathrm{d}Q_r^T}{\mathrm{d}r} + \nu Q_r^T = 0 \tag{69a}$$

$$w_0^T = 0 \tag{69b}$$

Free edge:

$$M_{rr}^T = 0, \qquad P_{rr}^T = 0, \qquad \text{which imply} \quad r\frac{\mathrm{d}Q_r^T}{\mathrm{d}r} + \nu Q_r^T = 0 \tag{70a}$$

$$rV_r^T = r\bar{Q}_r^T + \alpha r\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}r} = 0 \tag{70b}$$

For solid circular plates, we have the additional boundary conditions at the center of the plate, i.e., at $r = 0$:

$$Q_r^T = 0, \qquad \phi^T = 0, \qquad \frac{\mathrm{d}w_0^T}{\mathrm{d}r} = 0$$
$$rV_r^T = r\bar{Q}_r^T + \alpha r\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}r} = 0 \tag{71}$$

For annular plates, the boundary conditions at the inner edge are given by the type of edge support there.

D. Example

Here we present an example to illustrate the derivation of the solutions of the third-order theory using the relationships developed in the present paper between CPT and TPT. First note that Eq. (63) can be expressed in the alternative form

$$\frac{\mathrm{d}^2 Q_r^T}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}Q_r^T}{\mathrm{d}r} - \left(\frac{1}{r^2} + \xi_1\right)Q_r^T = -\xi_2\left(Q_r^C + \frac{c_1}{r}\right) \tag{72}$$

where

$$\xi_1 = \frac{D\hat{A}_{44}}{\alpha^2(DH - F^2)} = \frac{420(1-\nu)}{h^2} \tag{73a}$$

$$\xi_2 = \frac{\bar{D}\bar{A}_{44}}{\alpha^2(DH - F^2)} = \frac{420(1-\nu)}{h^2} \tag{73b}$$

The solution to the homogeneous differential equation

$$\frac{\mathrm{d}^2 Q_r^T}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}Q_r^T}{\mathrm{d}r} - \left(\frac{1}{r^2} + \xi_1\right)Q_r^T = 0 \tag{74}$$

is given by

$$Q_r^T(r) = c_5 I_1(\sqrt{\xi_1}r) + c_6 K_1(\sqrt{\xi_1}r) \tag{75}$$

where I_1 and K_1 are the first-order modified Bessel functions of the first and second kind, respectively.

Consider a solid circular plate under uniformly distributed load of intensity q_0 and clamped at the edge. For this case, the boundary conditions at $r = R_0$ give $Q_r^T(R_0)$ and $c_2 = 0$, and those at $r = 0$ give $Q_r^T(0) = 0$ and $c_1 = c_3 = 0$. Then the general solution to Eq. (72) is given by ($\xi_1 = \xi_2 \equiv \xi$)

$$Q_r^T(r) = c_5 I_1(\sqrt{\xi}r) + c_6 K_1(\sqrt{\xi}r) - (q_0 r/2) \tag{76}$$

Using the boundary conditions on Q_r^T, we obtain

$$c_6 = 0, \qquad c_5 = \frac{q_0 R_0}{2I_1(\sqrt{\xi}R_0)} \tag{77}$$

Hence the solution becomes

$$Q_r^T(r) = \frac{q_0 R_0}{2}\left[\frac{I_1(\sqrt{\xi}r)}{I_1(\sqrt{\xi}R_0)} - \frac{r}{R_0}\right] \tag{78a}$$

$$\int Q_r^T\,\mathrm{d}r = \left(\frac{q_0 R_0^2}{4}\right)\left[\frac{2I_0(\sqrt{\xi}r)}{R_0 I_1(\sqrt{\xi}R_0)\sqrt{\xi}} - \left(\frac{r}{R_0}\right)^2\right] \tag{78b}$$

Then the exact deflection of the TPT plate is given by

$$w_0^T(r) = w_0^C(r) + \left(\frac{\bar{D}}{D\bar{A}_{44}}\right)\left(\frac{q_0 R_0^2}{4}\right) \times \left[\frac{2I_0(\sqrt{\xi}r)}{R_0 I_1(\sqrt{\xi}R_0)\sqrt{\xi}} - \left(\frac{r}{R_0}\right)^2\right] - \frac{c_4}{D} \tag{79}$$

where the constant c_4 is evaluated using the boundary conditions $w_0^T = w_0^C = 0$ at $r = R_0$:

$$c_4 = \left(\frac{\bar{D}}{\bar{A}_{44}}\right)\left(\frac{q_0 R_0^2}{4}\right)\left[\frac{2I_0(\sqrt{\xi}R_0)}{R_0 I_1(\sqrt{\xi}R_0)\sqrt{\xi}} - 1\right] \tag{80}$$

Note that the deflection $w_0^C(r)$ of the classical plate theory for the problem is given by Eq. (44a). The maximum deflection is

$$w_{\max}^T = w_0^T(0) = \frac{q_0 R_0^4}{64D} + \frac{q_0 R_0^2 h^2}{20D(1-\nu)} \tag{81}$$

Comparing $w_{\max}^T$ with $w_{\max}^F$ from Eq. (45), we note that for $K_s = \frac{5}{6}$ the maximum deflection predicted by the first-order shear deformation theory coincides with that predicted by the third-order plate theory. Of course, the third-order theory does not require a shear correction coefficient. Further, the comparison of the solutions of the first-order and third-order theories for different boundary conditions and loadings may lead to different shear correction factors.

IV. Derivation of Finite Element Stiffness Matrix

The relationships presented earlier can be used to develop finite element models for axisymmetric bending of isotropic circular plates that contain the classical plate element as a special case. Such elements were developed for beams and axisymmetric bending of circular plates by Reddy[11] and Reddy et al.[12] For the sake of completeness, the procedure is illustrated for FST, and we leave similar development for TST as an exercise to the readers.

The general solution of Eqs. (20a) and (20b) for the isotropic case is [see Eqs. (38) and (40)]

$$w_0^F(r) = (1/4D)\{[(4D/GAK_s)\log r - r^2(\log r - 1)]c_1 - c_2 r^2 - 4c_3 \log r - 4c_4\}$$
$$= [\hat{c}_1 + \hat{c}_2 r^2 + \hat{c}_3 \log r + \hat{c}_4 r^2 \log r] \tag{82}$$

$$\phi^F(r) = (1/4D)[c_1 r(2\log r - 1) + 2c_2 r + 4(c_3/r)]$$
$$= -2\hat{c}_2 r - (\hat{c}_3/r) - \hat{c}_4[r(1 + 2\log r) + (1/r)\Gamma] \tag{83}$$

where $D = Eh^3/12(1-\nu^2)$, $\Gamma = (4D/GAK_s)$, and c_i are constants of integration. The classical plate theory solution is obtained from Eqs. (82) and (83) by setting $\Gamma = 0$. We shall use the preceding equations to construct the finite element model of the first-order shear deformation theory.

Consider a typical finite element located between $r_a \leq r \leq r_b$. Let the generalized displacements at nodes 1 and 2 of the element be defined as

$$w_0(r_a) = \Delta_1, \qquad \phi(r_a) \equiv \Delta_2$$
$$w_0(r_b) = \Delta_3, \qquad \phi(r_b) \equiv \Delta_4 \tag{84}$$

where ϕ denotes the slope (positive clockwise), which has different meaning in different theories, as defined next:

$$\phi = \begin{cases} -\dfrac{\mathrm{d}w_0}{\mathrm{d}r} & \text{for} \quad \text{CPT} \\ \phi & \text{for} \quad \text{FST} \end{cases} \tag{85}$$

Next, let Q_1 and Q_3 denote the shear forces (i.e., values of rQ_r) at nodes 1 and 2, respectively, and Q_2 and Q_4 the bending moments, i.e., values of rM_{rr}, at nodes 1 and 2, respectively.

Equations (82) and (83) can be expressed, with the definitions in Eqs. (84) and (85), as

$$\begin{Bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \end{Bmatrix} = \begin{bmatrix} 1 & r_a^2 & \log r_a & r_a^2 \log r_a \\ 0 & -2r_a & -(1/r_a) & -r_a(1 + 2\log r_a) - (1/r_a)\Gamma \\ 1 & r_b^2 & \log r_b & r_b^2 \log r_b \\ 0 & -2r_b & -(1/r_b) & -r_b(1 + 2\log r_b) - (1/r_b)\Gamma \end{bmatrix} \begin{Bmatrix} \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \\ \hat{c}_4 \end{Bmatrix}$$

or

$$\{\Delta\} = [H]\{\hat{c}\} \tag{86}$$

The nodal forces are

$$Q_1 \equiv -2\pi\left(rQ_r^F\right)_{r=r_a} = 8\pi D\hat{c}_4$$

$$Q_2 \equiv 2\pi\left(-rM_{rr}^F\right)_{r=r_a} = 2\pi D\left\{2(1+\nu)r_a\hat{c}_2 - \frac{(1-\nu)}{r_a}\hat{c}_3 + \left[\Lambda_a - \frac{(1-\nu)}{r_a}\Gamma\right]\hat{c}_4\right\}$$

$$Q_3 \equiv 2\pi\left(rQ_r^F\right)_{r=r_b} = -8\pi D\hat{c}_4$$

$$Q_4 \equiv 2\pi\left(rM_{rr}^F\right)_{r=r_b} = -2\pi D\left\{2(1+\nu)r_b\hat{c}_2 - \frac{(1-\nu)}{r_b}\hat{c}_3 + \left[\Lambda_b - \frac{(1-\nu)}{r_b}\Gamma\right]\hat{c}_4\right\}$$

or

$$\{Q\} = [G]\{\hat{c}\} = [G][H]^{-1}\{\Delta\} \equiv [K]\{\Delta\} \tag{87}$$

where

$$\Lambda_a = [2(1+\nu)\log r_a + (3+\nu)]r_a$$
$$\Lambda_b = [2(1+\nu)\log r_b + (3+\nu)]r_b \tag{88a}$$

$$[G] = 2\pi D \begin{bmatrix} 0 & 0 & 0 & 4 \\ 0 & 2(1+\nu)r_a & -\dfrac{(1-\nu)}{r_a} & \left[\Lambda_a - \dfrac{(1-\nu)}{r_a}\Gamma\right] \\ 0 & 0 & 0 & -4 \\ 0 & -2(1+\nu)r_b & \dfrac{(1-\nu)}{r_b} & -\left[\Lambda_b - \dfrac{(1-\nu)}{r_b}\Gamma\right] \end{bmatrix} \tag{88b}$$

and $[K] = [G][H]^{-1}$ is the element stiffness matrix. The stiffness matrix of the classical plate theory is obtained from $[K]$ by setting $\Gamma = 0$. The stiffness matrix derived here differs from the conventional Hermite cubic polynomial-based stiffness matrix,[13] and the present element gives exact nodal values and exhibits no locking.

V. Conclusions

In this paper, exact relationships between the bending solutions of the CPT and Reddy's TPT are developed. Because TPT is a sixth-order theory and CPT is a fourth-order theory, the exact relationships between deflections, slopes, moments, and shear forces of the two theories can only be developed by solving an additional second-order differential equation. In this paper, the second-order differential equation in terms of the transverse shear force Q_r is developed. Upon having the solution of this equation, the exact relationships between the deflections, slopes, bending moments, and shear forces of the two theories can be established.

The relationships developed herein facilitate actual derivation of the exact solutions of the TPT whenever the corresponding CPT results are available. It is also possible to develop finite element models of FST and TPT using the finite element model of CPT, as was illustrated herein for FST. The stiffness matrix of the shear deformable elements are also 4×4 for the pure bending case, and the finite elements are free from shear locking phenomenon[2,11,12] experienced by the conventional shear deformable finite elements. Finally, it is possible to develop the shear correction factors required in the first-order theory using the relationships between CPT, FST, and TPT. Such factors may depend on the boundary conditions as well as the applied transverse loads.

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